Equitable Colorings of Complete Tripartite Graphs

Dominic Ditizio, Bailey Orehosky, Maxwell Vlam

MA490 Mathematics Capstone Emily Marshall
Arcadia University Glenside, Pennsylvania 19038 USA

December 17, 2021

Abstract

In this paper, we prove the Equitable Coloring Conjecture for variations of complete tripartite graphs. Let G be a complete tripartite graph. When $G = K_{n,n,n}, \chi_=(G) = 3$. When $G = K_{n,n,2n}$, and when $n = 1, \chi_=(G) = 3$. When $n > 1, \chi_=(G) = 4$. When $G = K_{n,n+2,n+4}$, and $n < 5$ and n is odd, $\chi_=(G) = 6$ and when n is even, $\chi_=(G) = 5$. When $G = K_{n,n,n+2}$ and $1 \leq n \leq 4$, $\chi_=(G) = 4$. When $n = 5, 7$, $\chi_=(G) = 7$. When $n \geq 6$ and n is even, $\chi_=(G) = 6$. When $n \geq 9$ and n is odd and either n or $n + 2$ is a multiple of three, $\chi_=(G) = 9$.

1 Introduction

In this paper, all graphs are complete tripartite graphs. A *graph* is a mathematical object that consists of a set of *vertices* connected to one another by *edges.* A graph is considered *a complete graph* if all vertices are connected to each other with edges. *Independent sets* are a group of vertices in which no two are adjacent. If a graph consists of two or more independent sets of vertices, with the vertices not connected to any other in their set, it is considered a *multi-partite graph.* As we are focusing on *tripartite graphs*, it is important to define them in their own right. The definition is a set of vertices separated into three different partite sets, with no two vertices in the same set being adjacent.

To make a tripartite graph into a *complete tripartite graph*, each partite set must be connected to every vertex in the other partite sets. A vertex u in set one is not adjacent to a vertex v in the same set. To make a vertex adjacent to another, you only need to have an edge connecting the two. In general, multi-partite graphs each contain an *end vertex*, which is a vertex of degree one, provided that the set is greater than one. The sets of the vertices are referred to as *partite sets.*

The *degree* of a vertex is the amount of edges connected to a vertex in a graph G, with $\delta(G)$ denoting the smallest amount of edges and $\Delta(G)$ denoting the largest amount. The degree is often mentioned in theorems relating to the Equitable Coloring Conjecture, so it is important to know what the maximum degree is in regards to that.

2 Background

We will attempt to prove aspects of the Equitable Coloring Conjecture for complete tripartite graphs. The conjecture reads as follows:

Conjecture 2.1 (Equitable Coloring Conjecture)**.** *Let G be a connected graph. $\chi_=(G) \leq \Delta(G)$ unless G is an odd cycle or a complete graph [2].*

This conjecture focuses on *vertex colorings* and the *chromatic number*, which is denoted by $\chi(G)$. Vertex coloring is the act of labeling all vertices of a graph with different colors. A *proper coloring* is a graph coloring such that no two adjacent vertices share a color. $\chi(G)$ is the minimal number of colors given to the total amount of vertices. We wish to achieve an *equitable coloring* for complete tripartite graphs. An *equitable coloring* is a proper vertex coloring in which the sizes of the color classes differ by at most one, with the vertices being in groups of n and either $n+1$, or $n-1$ vertices. For this, we make our $\chi(G)$ into $\chi_=(G)$, which is the symbol for the *equitable chromatic number*. It serves the same purpose as the chromatic number, but it deals with equitable colorings rather than regular colorings.

Note; for all complete partite graphs, each vertex in a partite set is able to receive the same color because there are no edges between vertices within partite sets. No vertices have an adjacent vertex within its own partite set allowing them to all have the same color. We note this as it makes having a smaller chromatic number more likely than if the vertices were adjacent to one another.

The theorems we are going to refer to during this time include two written by Ko-Wei Lih and Pou-Lin Wu. Much of Lih and Wu's work came from the tremendous leaps taken by Walter Meyer as he proved parts of the Equitable Coloring Conjecture for trees and provided a basis for mathematicians to use in the future when working with the Equitable Coloring Conjecture [4]. Lih and Wu's theorems read:

Theorem 2.2. *Let $G = G(X, Y)$ be a connected bipartite graph. If G is different from any complete bipartite graph $K_{n,n}$, then G can be equitably colored with $\Delta(G)$ colors[3].*

and

Theorem 2.3. *The complete bipartite graph $K_{n,n}$ can be equitably colored with k colors if and only if $[n/[k/2]] - [n/[k/2]] \leq 1$ [3].*

These theorems relate directly to the base case we used to establish our theorems. If they worked for $K_{n,n}$, then they should also work for $K_{n,n,n}$. We applied what they learned to our graphs to further our research.

Another theorem that greatly impacted us was Brooks' Theorem, which stated:

Theorem 2.4 (Brooks' Theorem)**.** *Let G be a connected graph. If G is neither a complete graph or a cycle of odd length then $\chi(G) \leq \Delta(G)$. If G is either a complete graph or a cycle of odd length then $\chi(G) \leq \Delta(G) + 1$ [3].*

Brooks' Theorem gave a basis for cases that are not complete graphs or odd cycles. Despite our tripartite graphs being complete, they are not the same as a complete graph such as K_n. Those graphs are fully connected with no sets dividing them, so our chromatic number should be less than $\Delta(G)$ rather than $\Delta(G) + 1$.

As we move onto tripartite graphs we use the theorem from Wang and Zhang within the paper written by T. Karthick. The theorem states as follows:

Theorem 2.5. *If G is a complete multipartite graph different from $K_m (m \geq 1)$, then $\chi_=(G) \leq \Delta(G)$ [1], [5].*

This theorem allowed us to find a basis for finding the Equitable Coloring Conjecture in complete tripartite graphs. Using these theorems we further ventured into variations of complete tripartite graphs and were able to find proofs for four different types of tripartite graphs.

3 Our Theorems

Theorem 3.1. *For a complete tripartite graph $K_{n,n,n}, \chi_=(K_{n,n,n}) = 3$.*

Proof. For a graph $K_{n,n,n}$, we have three partite sets of n vertices. Because of this, we are allowed to make our color classes size n. Since each is the same value, we do not need to worry about it differing by more than one. As a result $K_{n,n,n}$ can be equitably colored with three colors, $\chi_=(K_{n,n,n}) = 3$. It can not be colored with less than three colors because then adjacent vertices would have to share a color and void the definition of both a proper coloring and an equitable coloring. □

Theorem 3.2. *For the complete tripartite graph $K_{n,n,2n}$, when $n = 1, \chi_=(K_{n,n,2n}) = 3$. When $n > 1, \chi_=(K_{n,n,2n}) = 4$.*

Proof. Case 1: When n is equal to one the third set of $2n$ vertices is equal only to two which differs from the first two set only by one which means you only need one color for the third set.

Case 2: The first two sets of vertices will each have one color per set containing n vertices each, and then the third set contains double number of vertices as the first two sets, so $2n$ is able to be divided into two sets of colors each containing n vertices and the total colors being four. It can not be lower than four because the color of $2n$ would have twice as many vertices as the color of sets one and two and this would not be equitably colored. □

Theorem 3.3. *For a complete tripartite graph $K_{n,n+2,n+4}$, with $n < 5$ and odd, $\chi_=(K_{n,n+2,n+4}) = 6$. For $n < 5$ and even, $\chi_=(K_{n,n+2,n+4}) = 5$.*

Proof. In both cases we establish that when n is closer to one, the other color class will contain $n + 1$ vertices as you are subtracting from the value, it will either produce a zero or a number that would cause more than the amount of colors established. When n gets closer to five, the second class will be $n - 1$, as adding one will not allow the vertex sets to divide equally between the two numbers.

Case 1: For $n < 5$ and odd, we can make our first partite set all one color with n vertices. From there, our second set will have one set of n vertices and one set of either $n+1$ or $n-1$ vertices. Our third partite set will have one group of n vertices and two groups of either $n + 1$ or $n - 1$ vertices. With this grouping, we will have a chromatic number equal to six.

Case 2: For $n < 5$ and even, our first partite set will have n vertices of one color. If $n = 2$, the layout of the second set will be two groups of n vertices. The third will be two groups of $n + 1$ vertices. If $n = 4$, the second set will have two groups of $n - 1$ vertices, and the third set will be be two groups of n vertices. □

Theorem 3.4. *For the complete tripartite graph $K_{n,n,n+2}$, the equitable chromatic number is based on the value of vertices in each partite set. When $n \leq 4$, $\chi_=(K_{n,n,n+2}) = 4$. When $n \geq 6$ and even, $\chi_=(K_{n,n,n+2}) = 6$. When $n = 5, 7$, $\chi_=(K_{n,n,n+2}) = 7$. And when $n \geq 9$ and odd with either n or $n + 2$ being a multiple of three, $\chi_=(K_{n,n,n+2}) = 9$.*

Proof. Case 1: Let G be the complete tripartite graph $K_{n,n,n+2}$ and $1 \leq n \leq 4$. The first two partite sets with n vertices each will have one color. For the third partite set of $n + 2$ vertices, we are able to use two colors because we can split $n + 2$ into two sets of either two sets of n vertices or n and $n + 1$ vertices which is within one of n making $\chi_=(G) = 4$. For all four values of $n, \chi_=(G) \leq \Delta(G)$ which follows the definition of an equitable coloring. $\chi_=(G)$ is never able to be less than four because the third partite set of $n + 2$ is always two greater then the first two partite sets of n vertices and that would not be an equitable coloring.

Case 2: Let G be the complete tripartite graph $K_{n,n,n+2}$ and $n = 5$ or 7. The first two partite sets are colored with two colors each. To do this we divide the partite set into two subsets each with vertices that differ by at most one. For $n = 5$ we divide the first two partite sets into subsets of 2 and 3 vertices. For $n = 7$ we divide the first two partite sets into subsets of 4 and 3 vertices. Then for the third partite set, we divide it into three subsets of colors, with each subset containing the same number of vertices as one of the two subsets in the first two partite sets. When $n = 5$ the third partite set of $n+2$ vertices is divided into two subsets of two and one subset of three vertices. When $n = 7$ the third partite set of $n+2$ vertices is divided into three subsets of each with three vertices. This brings the equitable chromatic number to $\chi_=(G) = 7$ for both $n = 5$ and 7. The chromatic number can not be any lower than seven because, either the number of vertices in one color would have to be distributed into two partite sets making it not a proper coloring, or because the number of vertices in each color would differ by more than one thus no longer be equitably colored.

Case 3: Let G be the complete tripartite graph $K_{n,n,n+2}$ when $n \geq 6$ and n is an even integer. Since n is an even integer, $n+2$ is also an even integer. This allows both n and $n+2$ to be able to be divided by two and create two even subsets of vertices within each of the three partite sets. Because $n+2$ is always two greater than n, $\frac{n+2}{2}$ is always one more than $\frac{n}{2}$. This means that each of the six subsets of vertices has either $\frac{n}{2}$ or $\frac{n}{2}+1$ vertices allowing the graph to have the equitable chromatic number $\chi_=(G) = 6$, with all colors having a number vertices within one of each other. For all even values of n when $n \geq 6, \chi_=(G) \leq \Delta(G)$ which follows the definition of an equitable coloring. $\chi_=(G)$ is never able to be less than six because of two reasons. One is that one color would be split between two partite sets and then have adjacent vertices with the same color which is not a proper coloring. The second reason is because if one partite set is colored with one color, then that one color will have a difference of more than one from the rest of the colors making it no longer equitably colored.

Case 4: Let G be the complete tripartite graph $K_{n,n,n+2}$. Also, let n be an odd integer, with $n \geq 9$, and let n or $n+2$ be a multiple of three. This allows all three partite sets to be divided into three subsets of vertices. When n is a multiple of three, the first two partite sets will each have three subsets of vertices, containing $\frac{n}{3}$ vertices each. This allows the third partite set to all be divided into three subsets of vertices, with either $\frac{n}{3}$ or $\frac{n}{3}+1$ vertices each. This is the same case for the $n+2$ is a multiple of three except, the third partite set will have the three subsets with $\frac{n+2}{3}$ vertices and the two partite sets with n will then have three subsets of $\frac{n+2}{3}$ or $\frac{n+2}{3}-1$ vertices each, allowing $\chi_=(G) = 9$ and $\chi_=(G) \leq \Delta(G)$ to follow the definition of an equitable coloring. $\chi_=(G) = 9$ is not able to be less than nine because one color class would have a difference of more then one vertices from the other color classes, voiding the definition of equitable colorings. This would also have adjacent vertices with the same color thus voiding the definition of proper coloring. □

4 Conclusion

In conclusion, four theorems were found and proved for complete tripartite graphs and the Equitable Coloring Conjecture. We studied up on our chosen topic by reading through a number of papers by Ko-Wei Lih and Pou-Lin Wu in order to get background information on bipartite graphs and to give us some possible ideas on tripartite graphs. Next we worked on tripartite graphs directly. We tried out different forms of tripartite graphs and discovered patterns for specific cases. By writing out a number of different graphs we managed to make steps forward. After working on graphs and figuring out how they might work we rewrote our findings into theorems. We chose a few of the more extensive graphs that were not so easily explained and sought to give them a proper explanation. We concluded the best course for proving our theorems would be directly. As a group we managed to construct a well reasoned proof for each of our four theorems. We had started with a base idea of where to start and have taken our work to its conclusion.

If we had more time to work on equitable colorings of complete tripartite graphs we would begin to make connections between our findings and real world applications. We would also liked to have explored the uses of algorithms and how they could possibly order our work for different variations of complete tripartite graphs. In the end, conclusions were drawn on complete tripartite graphs through research and collaboration leading to small advancements in equitable colorings of complete tripartite graphs.

5 Acknowledgements

The authors are grateful to Doctor Emily Marshall for helping them pinpoint a topic and discussing the applications of theorems in regards to our research. We would also like to thank Doctor Carlos Ortiz for providing support in regards to writing proofs and the use of LaTeX.

References

[1] T. Karthick. Note on equitable coloring of graphs. *AUSTRALASIAN JOURNAL OF COMBINATORICS*, 59(2):551–559, 2014.

[2] Ko-Wei Lih. The Equitable Coloring of Graphs. *HANDBOOK OF COMBINATORIAL OPTIMIZATION*, 3(2):543–566, 1998.

[3] Ko-Wei Lih and Pou-Lin Wu. On Equitable Coloring of Bipartite Graphs. *Discrete Mathematics 1*, 151:150–160, 1996.

[4] Walter Meyer. Equitable Coloring. *Mathematical Association of America*, 80(8):920–922, 1973.

[5] W. Wang and K. Zhang. Equitable colorings of line graphs and complete r-partite graphs. *Systems Science and Math*, 13:190–194, 2000.