Observing Influenza A Through the Basis of Discretized Fractional Derivatives in SIR Models

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Abstract

We begin by understanding the fundamentals of fractional calculus and how it is used as a basis when applied to SIR models. In this particular case, we shall use fractional order of discretized differential equations for our SIR Model to examine the biological processes of the epidemic by Influenza A.

Introduction

Understanding fractional calculus is the quintessential preliminary to understanding how Susceptibility, Infected, and Recovered (SIR) models are analyzed and presented with their statistical data. Another main objective is to understand SIR models in the case of fractional derivatives. We analyze the basic rudiments (properties) of fractional calculus because in the article, "On a Discretized Fractional-Order SIR Model for Influenza A Viruses", they are utilizing fractional derivatives to observe a system of differential equations of fractional derivative-order, which includes memory effect and fractional-order parameters. We must first understand the basic properties of fractional calculus, which we will discuss after defining what a fractional derivative and integral is.

Intro to Fractional Calculus

Fractional Derivatives

Example: Take the basic concept of derivation of any order for an exponential function

$$D^n e^{ax} = a^n e^{ax}$$

This suggests that $D^\alpha e^{ax} = a^\alpha e^{ax}$ where α could be any positive real number.

- When α is a negative integer, it is reasonable to deduce that it represents an n interated integral

  ex. $D^{-1}f(x) = \int f(x)dx$

- If α is a positive real number, then D^α represents a derivative.

What if we want to define fractional derivatives and integral for other functions different from e^x? For example, what could be the half derivative of x^p? First, the general expression of taking the derivative of x^p to the nth order is as follows:

$$D^n x^p = \frac{p!}{(p-n)!} x^{p-n}$$

Key observation: the factorials can be represented by using the Gamma function.
The definition of a Gamma function is as follows: A Gamma function, see [2], is used to represent $p!$ and $(p-n)!$ where p and n are non-natural orders. In essence, Gamma functions are used to represent factorials of non-natural order and by definition is as follows:

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$$

where some of the major key properties of a Gamma function include but are not limited to the following see [2]:

- $\Gamma(z+1) = z\Gamma(z)$
- $\Gamma(n) = (n-1)!, n \in \mathbb{N}$
- $\Gamma(1/2) = \sqrt{\pi}$

So since we know the properties of the Gamma function, we can rewrite the fractional derivative of a power function to become

$$D^n x^p = \frac{\Gamma(p+1)x^{p-n}}{\Gamma(p-n+1)}$$

and since n is a non-integer, we can put

$$D^\alpha x^p = \frac{\Gamma(p+1)x^{p-\alpha}}{\Gamma(p-\alpha+1)}$$

for any α.

The above equation then can be extended for a large number of functions where given any function, it can be expanded using a Taylor series, see [4],

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

and by assuming we can differentiate each term, we get

$$D^\alpha f(x) = \sum_{n=0}^{\infty} a_n D^\alpha x^n = \sum_{n=0}^{\infty} a_n \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)} x^{n-\alpha}$$

There are different definitions of fractional derivatives contingent upon the different types of functions. For example, the definition of the fractional derivative of a trig function will be different. We know the pattern of the derivative of trig functions goes as follows see [5]:

$$D^0 sin(x) = sin(x), D^1 sin(x) = cos(x), D^2 sin(x) = -sin(x)...$$

This does not present an obvious pattern to find $D^{1/2} sin(x)$. However we'll notice an interesting feature by graphing the function. Each time we differentiate the graph of sin(x) it is being shifted to the left by $\pi/2$. Thus differentiating sin(x) n times will shift the graph to the left by the quantity of $n\pi/2$. With that information we can conclude that the fractional derivative definition of trig functions will be see [5]:

$$D^\alpha sin(x) = sin(x + \frac{\alpha\pi}{2})$$
$$D^\alpha cos(x) = cos(x + \frac{\alpha\pi}{2})$$

We can conclude that there are different definitions of fractional derivatives for different types of functions.

Fractional Integrals

After examining fractional derivatives, we will now consider fractional integrals and examine how they are defined. Fractional integrals are defined as $D^{-n} f(t)$ where $n > 0$ and n is an integer. As we stated earlier, an example is as follows: $D^{-1} f(t) = \int f(t)dt$ when $n = 1$. Now, we let I be the closed interval $[0, X]$ where f is continuous on I. Since n is of non-natural order, we shall call it β where I^β is the fractional integer order in $\mathbb{R}^+$ of the function $f(t)$, defined as:

$$I^\beta f(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f(s)ds$$

This definition works because if we were to plug in a certain value for β, the definition will hold. For example, when $\beta = 2$,

$$I^2 f(t) = \int_0^t \frac{(t-s)^{2-1}}{\Gamma(2)} f(s)ds$$

this becomes

$$I^2 f(t) = \frac{1}{\Gamma(2)} \int_0^t (t-s) f(s)ds$$

which simplifies to

$$I^2 f(t) = \frac{1}{2} \int_0^t (t-s) f(s)ds$$

We can show that this is true using Cauchy's formula for repeated integration where it is (see [3])

$$I^n f(t) = \int_a^t \int_a^\tau \ldots \int_a^{\tau_{n-1}} f(\tau)d\tau \ldots d\tau_2 d\tau_1 = \frac{1}{(n-1)!} \int_a^t (t-\tau)^{n-1} f(\tau)d\tau$$

which we know the $(n-1)!$ is the Gamma function so

$$I^n(t) = \frac{1}{\Gamma(n)} \int_a^t (t-\tau)^{n-1} f(\tau)d\tau$$

Thus:

$$I^2 f(t) = \frac{1}{\Gamma(2)} \int_0^t (t-s)^{2-1} f(s)ds = \int_0^t \int_0^\tau f(\tau)d\tau dt$$

In essence, this is a fractional integral to the first order, which complies with the definition of a fractional integral, justified by also using Cauchy's formula for repeated integration.

Properties of Fractional Calculus

We selected the Caputo fractional derivative because the initial value of fractional differential equation with the Caputo derivative is the same as the initial value of the integer differential equation. There is a difference between fractional differential equations and normal differential equations in which one lacks "memory". Normal differential equations have "local" conditions whereas fractional differential equations have "memory". This is the reason why one prefers fractional derivatives to standard derivatives when modeling biological processes.

The Caputo definition are:

$$I_a^\beta f(x) = \int_a^x \frac{(x-s)^{\beta-1}}{\Gamma(\beta)} f(s) ds$$

and the definition of a fractional derivative α order is:

$$D_a^\alpha f(x) = \int_a^x \frac{(x-s)^{-\alpha}}{\Gamma(1-\alpha)} f'(s) ds$$

For our use of fractional calculus, we will need to understand the main principles of this branch of mathematics. For these main principles let $\beta, \gamma \in \mathbb{R}^+, \alpha \in (0,1)$. The main principles are:

1. $I_b^a : L^1 \to L^1$, and if $f(x) \in L^1$, then $I_a^\gamma I_a^\beta f(x) = I_a^{\gamma+\beta} f(x)$.
2. $\lim_{\beta \to n} I_a^\beta f(x) = I_a^n f(x)$ uniformly on [a,b], n= 1,2,3,..., where $I_a^1 f(x) = \int_a^x f(s) ds$
3. If $f''(x)$ exists and is uniformly bounded by M in interval $[a,b]$, then, $\lim_{\alpha \to 1^{-1}} D_a^\alpha f(x) = \frac{df(x)}{dx}$

Property One

Before we prove property one we have to use Lemma 2.1.This Lemma is going to be a vital resource for the first proof of Caputo Fractional Derivatives. For us to use the Lemma we have to use a defintion of β that will be used later in proof one. The definition of the function β goes as follows

$$\beta(p,q) = \int_0^1 (1-s)^{p-1} s^{q-1} ds$$

$\beta(p,q)$ has the property:

$$\beta(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

Proof of this property can be found in [9]

Lemma 2.1 Let $0 \le \tau < t$ and $p > 0, q > 0$ then we have

$$\int_\tau^t (t-s)^{p-1}(s-\tau)^{q-1} ds = (t-\tau)^{p+q+1} \beta(p,q)$$

Proof. We can prove this Lemma with the process below. First we will have to change the order of integration from s to σ. This means we will derive the equation in respect to σ instead of s. This means we will rewrite s in the equation in terms of the definition: $s = \tau + \sigma(t-\tau)$.

$$\int_{\tau}^{t}(t-(\tau+\sigma(t-\tau)))^{p-1}((\tau+\sigma(t-\tau))-\tau)^{q-1}ds$$

When deriving in terms σ the ds of the equation will change. Such that $\frac{ds}{d\sigma} = (t-\tau) \longrightarrow ds = d\sigma(t-\tau)$

$$\int_{\tau}^{t}(t-(\tau+\sigma(t-\tau)))^{p-1}((\tau+\sigma(t-\tau))-\tau)^{q-1}(t-\tau)d\sigma$$

If you use the definition $s = \tau + \sigma(t-\tau)$ and replace s with τ, then $\sigma = 0$. When we replace s with t, then $\sigma = 1$. This would lead to this integral below:

$$\int_{0}^{1}((1-\sigma)(t-\tau))^{p-1}(\sigma(t-\tau))^{q-1}(t-\tau)d\sigma$$

$$= (t-\tau)^{p+q-1}\int_{0}^{1}(1-\sigma)^{p-1}\sigma^{q-1}d\sigma$$

$$= (t-\tau)^{p+q-1}\beta(p,q)$$

□

Theorem 1. *$I_b^a : L^1 \to L^1$, and if f(x) $\in L^1$, then $I_a^\gamma I_a^\beta f(x) = I_a^{\gamma+\beta} f(x)$.*

Proof. Direct Proof. Suppose $I_b^a : L^1 \to L^1$ and $f(x) \in L^1$.

By definition $I_a^\beta f(x)$ is

$$I_a^\beta f(x) = \int_a^x \frac{((x-s)^{\beta-1})}{(\Gamma(\beta))} f(s)ds \tag{1}$$

Since we stated this definition earlier in the paper. Now the expression $I_a^\gamma I_a^\beta f(x)$ can be written as a composite function such that we write $I_a^\gamma(I_a^\beta f(x))$.

$$I_a^\gamma(I_a^\beta f(x)) = \int_a^x \frac{((x-s)^{\beta-1})}{(\Gamma(\beta))}(I_a^\gamma f(s))ds \tag{2}$$

$$= \int_a^x \frac{((x-s)^{\beta-1})}{(\Gamma(\beta))}(\int_a^s \frac{((s-\epsilon)^{\gamma-1})}{\Gamma(\gamma)} f(\epsilon)d\epsilon)ds$$

For the next equation we'll need to use Fubini's Theorem to change the order of integration. Now Fubini's Theorem is a process of getting the result of a double integral using an iterated integral [9]. An example of this process can be written like this below

$$\int_x(\int_y f(x,y)dy)dx = \int_y(\int_x f(x,y)dx)dy$$

For Fubini's theorem, the order of integration changes therefore the bounds of the integral changes. Let's use figure 1 above as a guide through the integral

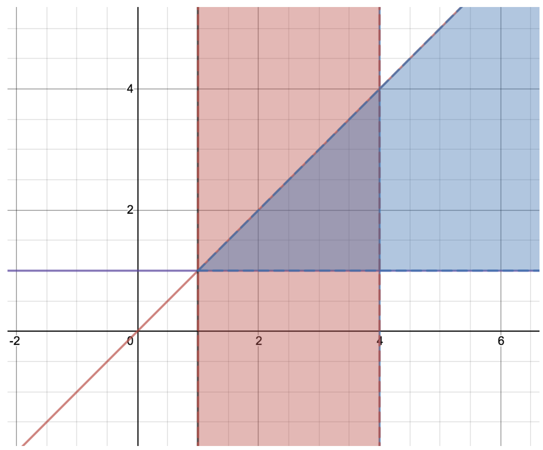


Figure 1:

change. The order of the initial function can be written as $\int_a^x \int_a^s f(s,\epsilon)d\epsilon ds$. a and x representing constants for functions of s and ϵ. In the Figure 1 graph, we represented a = 1 and x = 4. Graphically in Figure 1 ϵ represents the Y axis while s represents the x axis. Now for our initial integral we graph a = ϵ which would result in a horizontal line at ϵ = 1 as the starting point. Then s = ϵ would be a line identical to y=x on a normal graph. These two lines are represented in blue. For $\int_a^x$ ds a = s and x = s will result in two vertical lines as a and x are both constants as stated earlier. This is represented in red. Now we will change the order of integration, evaluating the intersection of both integrals graphically and determining the endpoints from the s direction, then the ϵ direction. In terms of ds, the initial point of the double integral would be ϵ from the s = ϵ line. The next endpoint would be s = x as that's where the integral would stop. Now in the ϵ direction, the lower endpoint would be ϵ = a. The upper endpoint would be the second intersection point of the lines s = ϵ and s = x. In terms ϵ, this value is 4 or x. With this evaluation our new integral bounds would be $\int_a^x \int_\epsilon^x$. This would lead to this statement below

$$\frac{1}{\Gamma(\beta)\Gamma(\gamma)} \int_a^x f(\epsilon)(\int_\epsilon^x (x-s)^{\beta-1}(x-\epsilon)^{\gamma-1}ds)d\epsilon \tag{3}$$

After using Fubini's theorem to change the order of integration we can use Lemma 2.1 as explained earlier to simplify the previous equation to

$$\frac{1}{\Gamma(\beta)\Gamma(\gamma)} \int_a^x f(\epsilon)(s-\epsilon)^{\beta+\gamma+1}\beta(\beta,\gamma)d\epsilon \tag{4}$$

After using Lemma 2.1 we can use the definition $\beta(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$ within the equation

$$\frac{1}{\Gamma(\beta)\Gamma(\gamma)} * \beta(\beta,\gamma) = \frac{1}{\Gamma(\beta)\Gamma(\gamma)} * \frac{\Gamma(\beta)\Gamma(\gamma)}{\Gamma(\beta+\gamma)} = \frac{1}{\Gamma(\beta+\gamma)}$$

After applying this property of β within our equation we will end up with

$$\frac{1}{\Gamma(\beta+\gamma)} \int_a^x (s-\epsilon)^{\beta+\gamma+1} f(\epsilon) d\epsilon \tag{5}$$

This previous equation is the equivalent of $I^{\beta+\gamma}$(f(x)) which is equal to $I_a^\gamma I_a^\beta f(x)$ □

Property Two

Theorem 2. $\lim_{\beta\to n} I_a^\beta f(x) = I_a^n f(x)$ *uniformly on [a,b], n= 1,2,3,..., where* $I_a^1 f(x) = \int_a^x f(s)ds$

Proof. We prove this theorem by direct proof. By definition:

$$I_a^\beta f(x) = \int_a^x \frac{(x-s)^{\beta-1}}{\Gamma(\beta)} f(s)ds$$

To prove $\lim_{\beta\to n} I_a^\beta f(x) = I_a^n f(x)$ we will pass the limit under the integral to attain the desired outcome as follows

$$\lim_{\beta\to n} I_a^\beta f(x) = \lim_{\beta\to n} \int_a^x \frac{(x-s)^{\beta-1}}{\Gamma(\beta)} f(s)ds$$

$$\lim_{\beta\to n} \frac{1}{\Gamma(\beta)} \int_a^x (x-s)^{\beta-1} f(s)ds$$

$$\frac{1}{\Gamma(n)} \lim_{\beta\to n} \int_a^x (x-s)^{\beta-1} f(s)ds \tag{6}$$

$$\frac{1}{\Gamma(n)} \int_a^x \lim_{\beta\to n} (x-s)^{\beta-1} f(s)ds \tag{7}$$

We will show how (6) and (7) is possible in full detail later

$$\frac{1}{\Gamma(n)} \int_a^x (x-s)^{n-1} f(s)ds = I_a^n f(x)$$

If we pass the limit under the integral, we are done with the proof of the second property. But the question is, when can we guarantee that in this particular case is valid and appropriate?

We can only pass the limit under the integral if the sequences of functions $(x-s)^{\beta-1} f(s) \to^{unif} (x-s)^{n-1} f(s)$ as $\beta \to n$ in other words, where uniform convergence takes place, see [10]. Since f is defined on $[a,b]$ and is L^1, then $f([a,b])$ is bounded by a number, we shall call it M. So $\forall s \in [a,b]$, $|f(s)| < M$.

We shall define uniform convergence for real-valued functions [10]. Informal definition: if f_n converges to f uniformly, then the rate at which $f_n(x)$ approaches $f(x)$ is uniform throughout throughout its domain is in this intuition: M is determined through guaranteeing that $f_n(x)$ falls a certain distance of ϵ of $f(x)$ where there is a single value $N = N(\epsilon)$ independent of x, where choosing the larger n of M will be sufficient.

The important aspect of uniform convergence is to know this key property: if $f_n \to^{unif} f$ then $\lim_{n\to\infty} \int f_n ds \to \int \lim_{n\to\infty} f_n ds$, see [10]. Then we must prove that $(x-s)^{\beta-1} f(s) \to^{unif} (x-s)^{n-1}$ as $\beta \to n$ so that we can use the key property in our proof.

We shall now show (6) and (7) is possible and appropriate to do by proving uniform convergence.

Proof of Uniform Convergence of the sequence $(x-s)^{\beta-1}f(s)$ **to** $(x-s)^{n-1}f(s)$

We can say that $\forall x \in [a,b]$ and $\forall s \in [a,x]$ $0 \leq (x-s)^n \leq (b-a)^n$. if we assume that $\beta > n$

Then $|(x-s)^{\beta-1} - (x-s)^{n-1}| = (x-s)^{n-1}|(x-s)^{\beta-n} - 1|$.
This it can be rewritten as $0 \leq |(x-s)^{n-1}f(s)| \leq (b-a)^{n-1}M$ so $|(x-s)^{\beta-1}f(s) - (x-s)^{n-1}f(s)| = |f(s)(x-s)^{n-1}||(x-s)^{\beta-n} - 1| \leq M(b-a)^{n-1}|(b-a)^{\beta-n} - 1|$. So, $\lim_{\beta\to n} |(x-s)^{\beta-1}f(s) - (x-s)^{n-1}f(s)| \leq M(b-a)^{n-1}|(b-a)^{\beta-n} - 1| = 0$.

Since the limit of this interval is equal to zero, then we prove that $(x-s)^{\beta-1}f(s) \to^{unif} (x-s)^{n-1}f(s)$ is uniformly convergent.

Since we know know that the sequence is uniformly continuous and convergent, this ultimately proves that we can pass the limit under the integral which proves $\lim_{\beta\to n} I_a^\beta f(x) = I_a^n f(x)$ □

Property Three

Theorem 3. *If $f''(x)$ exists and is uniformly bounded by M in interval $[a,b]$, then,* $\lim_{\alpha\to 1^{-1}} D_a^\alpha f(x) = \frac{df(x)}{dx}$

Proof. By direct proof.

In order to conclude

$$\lim_{\alpha\to 1} \int_a^x \frac{(x-s)^{-\alpha}}{\Gamma(1-\alpha)} f'(s)ds = \frac{df(x)}{dx}$$

we shall use integration by parts where

$u' = (x-s)^{-\alpha} \qquad v = f'(s)$

$u = \frac{-(x-s)^{1-\alpha}}{1-\alpha} \qquad v' = f''(s)$

where we obtain

$$\lim_{\alpha\to 1}(\frac{1}{\Gamma(1-\alpha)}\int_a^x (x-s)^{-\alpha} f'(s)ds) = \lim_{\alpha\to 1}(\frac{1}{\Gamma(1-\alpha)}[\frac{(x-s)^{1-\alpha}}{(1-\alpha)} f'(s)|_a^x - \int_a^x \frac{(x-a)^{1-\alpha}}{(1-\alpha)} f''(s)ds])$$

by evaluating $\frac{(x-s)^{1-\alpha}}{(1-\alpha)} f'(s)|_a^x$ we get $\frac{(x-x)^{1-\alpha}}{(1-\alpha)} f'(x) - \frac{(x-a)^{1-\alpha}}{(1-\alpha)} f'(a)$
so now we obtain

$$lim_{\alpha\to 1} \frac{-(x-a)^{1-\alpha} f'(a)}{(1-\alpha)\Gamma(1-\alpha)} - \frac{1}{(1-\alpha)\Gamma(1-\alpha)} \int_a^x (x-s)^{1-\alpha} f''(s)ds$$

One of the basic properties of a Gamma function is as follows: $\forall z \in \mathbb{R}$ $\Gamma(z+1) = z\Gamma(z)$ see [2]

By applying this property to our equation above, we now get

$$lim_{\alpha\to 1} \frac{-(x-a)^{1-\alpha} f'(a)}{\Gamma(2-\alpha)} - \frac{1}{\Gamma(2-\alpha)} \int_a^x (x-s)^{1-\alpha} f''(s)ds$$

$\lim_{\alpha\to 1} \Gamma(2-\alpha) = \Gamma(1) = 1$ where this yields the following result:

$$\frac{-(x-a)^{1-1} f'(a)}{(1)} - \frac{1}{(1)} \lim_{\alpha\to 1} \int_a^x (x-s)^{1-\alpha} f''(s)ds$$

which is simply

$$f'(a) + \lim_{\alpha\to 1} \int_a^x (x-s)^{1-\alpha} f''(s)ds \tag{8}$$

Now that we have acquired the equation above, we can pass the limit under the integral where

$$f'(a) + \int_a^x \lim_{\alpha\to 1} (x-s)^{1-\alpha} f''(s)ds \tag{9}$$

we will explain later how (8) and (9) is true with the proof of uniform convergence, which allows us to put the limit under the integral to be true

which then simplifies to

$$f'(a) + \int_a^x f"(s)ds$$

when evaluated we acquire $f'(s)|_a^x = f'(x) - f'(a)$ and finally, we have

$$f'(a) + [f'(x) - f'(a)] = f'(x)$$

But when can we guarantee to pass the limit under the integral? This action is appropriate only if we prove that the sequence of $(x-s)^{1-\alpha} f"(s) \to^{unif}$ to $f"(s)$ as the $\alpha \to 1$

Proof of Uniform Convergence

To prove uniform convergence, the sequences of functions $(x-s)^{1-\alpha}f"(s) \rightarrow^{unif} f"(s)$ as $\alpha \rightarrow 1$ see [10].

We know that $\forall x \in [a,b]$ and $\forall s \in [a,x]$ and $0 \leq \alpha < 1$ so we have

$$|(x-s)^{1-\alpha}f"(s) - f"(s)| \leq |f"(s)||(x-s)^{1-\alpha} - 1| \leq |M||(b-a)^{1-\alpha} - 1|$$

Because $f"(s)$ is bounded by M. We assume that $0 \leq \alpha < 1$ so that

$$lim_{\alpha \rightarrow 1^{-1}} |M||(b-a)^{1-\alpha} - 1| = 0$$

Since the limit of the interval is zero, we have proved uniform convergence where the series of functions $(x-s)^{1-\alpha}f"(s) \rightarrow^{unif} f"(s)$ as $\alpha \rightarrow 1$.

Then, we can conclude that we can pass the limit under the integral since we have proven that the sequences of functions $(x-s)^{1-\alpha}f"(s) \rightarrow^{unif} f"(s)$ which ultimately proves that $\lim_{\alpha \rightarrow 1} D_a^{\alpha} f(x) = \frac{df(x)}{dx}$

□

Formulation of SIR Models

Now to model the Influenza virus, we will use a SIR model. SIR is an acronym where S denotes the number for susceptibility, I for the number infected, and R stands for number of people recovered [7]. The initial equations associated with these values are

$$\frac{dS}{dt} = -\beta SI$$

$$\frac{dI}{dt} = \beta SI - \gamma I$$

$$\frac{dR}{dt} = \gamma I$$

The β is the rate of transmission (rate at which disease is transmitted from person to person) and the γ is the rate of recovery. Now why are these equations written like this. Lets start with the equation for recovery. If γ is representing rate of recovery then γI would represent the amount of people who have recovered from infection. For the infection equation, γI would be negative because the amount of people who were infected and recovered would be subtracted out of the equation. βSI would represent the amount of people who were susceptible that have been transmitted the disease. Essentially the newly infected. Lastly the Susceptible equation would be -βSI as the opposite represents the newly infected, this would represent the susceptible population that is not infected.

These sets of equations is know as the Kermack and McKendrick equations.[7] Let's let N denote the size of the population then obviously

$$N = S + I + R$$

$$N^{\cdot} = S^{\cdot} + I^{\cdot} + R^{\cdot} = 0$$

With the $\cdot$ representing derivatives with respect to time. The sum of the derivatives will equal 0. The reason why the sum of Derivatives of S, I and R are 0 is because the sum of the three quantities is constant instead of being functions of time. Therefore the derivatives of all of them will be 0.

One thing these previous equations didn't take into account is the rate of birth and death in population N. These next three equations take this into account using μ as the constant rate of birth.

$$\frac{dS}{dt} = \mu - \mu S - \beta SI$$

$$\frac{dI}{dt} = \beta SI - (\mu + \gamma)I$$

$$\frac{dR}{dt} = \gamma I - \mu R$$

These three equations above apply the constant birth rate with the assumption that it applies to the constant death rate as well.

Now we can introduce fractional order derivatives into our SIR equations. This is because these equations may do a better job at modeling an epidemic as fractional order derivatives have memory because memory is important in biological processes.

$$\frac{d^{\alpha}S}{dt} = \mu - \mu S(t) - \beta S(t)I(t)$$

$$\frac{d^{\alpha}I}{dt} = \beta S(t)I(t) - (\mu + \gamma)I(t)$$

$$\frac{d^{\alpha}R}{dt} = \gamma I(t) - \mu R(t)$$

Fractional differential equations are good to model because they contain memory, but because it is too difficult to calculate, we discretize the fractional equations to make them easier to solve.

$$\frac{d^{\alpha}S}{dt^{\alpha}} = \mu - \mu S(r[\frac{t}{r}]) - \beta S(r[\frac{t}{r}])I(r[\frac{t}{r}])$$

$$\frac{d^{\alpha}I}{dt^{\alpha}} = \beta S(r[\frac{t}{r}])I(r[\frac{t}{r}]) - (\mu + \gamma)I(r[\frac{t}{r}])$$

$$\frac{d^\alpha R}{dt^\alpha} = \gamma I(r[\frac{t}{r}]) - \mu R(r[\frac{t}{r}])$$

The symbol of [.] denotes a greatest integer function of the elements inside of it. The fraction $\frac{t}{r}$ includes t and the discrezitation step size of 0 < r < 1. $[\frac{t}{r}]$ would take this value to the greatest integer form.

Simplifying Discretization

We are going to start with an example of $\frac{d^\alpha R}{dt^\alpha}$ to show how fractional equations using discretization and can be solved

$$\frac{d^\alpha R}{dt^\alpha} = \gamma I(r[\frac{t}{r}]) - \mu R(r[\frac{t}{r}])$$

Above is our original equation for recovery rate equation. We'll make our R(0) = R_0, and our I(0) = I_0. We will use a step method having t ∈ [0,r) then $\frac{t}{r}$ ∈ [0,1). $[\frac{t}{r}]$ will equal 0 to get us a solution

$$\frac{d^\alpha R}{dt^\alpha} = \gamma I_0 - \mu R_0$$

So how can we use this differential equation into our initial recovery rate equation? By α integration we get

$$R_1(t) - R_0 = I^\alpha(\gamma I_0 - \mu R_0)$$

$$= R_0 + (\gamma I_0 - \mu R_0) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds$$

Evaluating integral $\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds$

$$\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds =$$

$$\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds = \frac{1}{\Gamma(\alpha)} - \frac{(t-s)^\alpha}{\alpha}]_0^t = \frac{t^\alpha}{\alpha\Gamma(\alpha)}$$

$\alpha\Gamma(\alpha)$ simplifies to $\Gamma(1+\alpha)$[2]. Now we can complete our equation of $R_1(t)$:

$$R_1(t) = R_0 + (\gamma I_0 - \mu R_0)\frac{t^\alpha}{\Gamma(1+\alpha)}$$

We can use this same method for R_2. In this case we will let t $\in [r, 2r)$ and $\frac{t}{r} \in [1, 2)$ and $[\frac{t}{r}] = 1$, so $r * [\frac{t}{r}] = r$ and

$$R_2(t) = R_1(r) + I_r^\alpha(\gamma I(r) - \mu R(r)) =$$

$$R_1(r) + (\gamma I(r) - \mu R(r)) \int_r^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds$$

Evaluating integral $\int_r^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds$

$$\int_r^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds =$$

$$\frac{1}{\Gamma(\alpha)} \int_r^t (t-s)^{\alpha-1} ds = \frac{1}{\Gamma(\alpha)} - \frac{(t-s)^{\alpha}}{\alpha}]_r^t = \frac{(t-r)^{\alpha}}{\alpha\Gamma(\alpha)}$$

Now we can complete our equation for R_2(t):

$$R_1(r) + (\gamma I(r) - \mu R(r)) \frac{(t-r)^{\alpha}}{\Gamma(1+\alpha)}$$

We can continue repeating this process to deduce it to a final solution. We will have t $\in$ [nr,(n+1)r) and $\frac{t}{r} \in$ [n,n+1) to get

$$R_{n+1}(t) = R_n(nr) + (\gamma I(nr) - \mu R(nr)) \frac{(t-(nr))^{\alpha}}{\Gamma(\alpha+1)}$$

We will have $t \longrightarrow (n+1)r$ to obtain the discretization below

$$R_{n+1}((n+1)r) = R_n(nr) + \frac{(((n+1)r) - (nr))^{\alpha}}{\Gamma(\alpha+1)} (\gamma I(nr) - \mu R(nr))$$

That is

$$R_{n+1} = R_n + \frac{r^{\alpha}}{\Gamma(\alpha+1)} (\gamma I_n - \mu R_n)$$

This discretization process was used to simplify our initial SIR model equations. Now let's use this process for our susceptibility equation and infection equation. Let's start with S

$$\frac{d^{\alpha} S}{dt^{\alpha}} = \mu - \mu S(r[\frac{t}{r}]) - \beta S(r[\frac{t}{r}]) I(r[\frac{t}{r}])$$

Above is our original equation for susceptibility equation. We'll make our $S(0) = S_0$. We will use a step method having t $\in$ [0,r) then $\frac{t}{r} \in$ [0,1). This would get us a solution of

$$\frac{d^{\alpha} S}{dt^{\alpha}} = \mu - \mu S_0 - \beta S_0 I_0$$

So how can we use this differential equation into our initial susceptibility rate equation?

$$S_1(t) - S_0 = I_r^{\alpha} (\mu - \mu S_0 - \beta S_0 I_0)$$

$$S_1 = S_0 + (\mu - \mu S_0 - \beta S_0 I_0) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds$$

$$S_1 = S_0 + (\mu - \mu S_0 - \beta S_0 I_0)\frac{t^\alpha}{\Gamma(1+\alpha)}$$

We can use this same method for S_2. In this case we will let t $\in [r, 2r)$ and $\frac{t}{r} \in [1, 2)$

$$S_2(t) = S_1(r) + I_r^\alpha(\mu - \mu S(r) - \beta S(r)I(r))$$

$$S_0(r) + (\mu - \mu S(r) - \beta S(r)I(r))\int_r^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}ds$$

$$= S_1(r) + (\mu - \mu S(r) - \beta S(r)I(r))\frac{(t-r)^\alpha}{\Gamma(1+\alpha)}$$

We can continue repeating this process to deduce it to a final solution. We will have t $\in$ [nr,(n+1)r) and $\frac{t}{r} \in$ [n,n+1)

$$S_{n+1}(t) = S_n(nr) + nr(\mu - \mu S - \beta SI)\frac{(t-(nr))^\alpha}{\Gamma(\alpha)}$$

We will have $t \to (n+1)r$ to obtain the discretization below

$$S_{n+1}((n+1)r) = S_n(nr) + \frac{(((n+1)r) - (nr))^\alpha}{\Gamma(\alpha+1)}(\mu - \mu S(nr) - \beta S(nr)I(nr))$$

That is

$$S_{n+1} = S_n + \frac{r^\alpha}{\Gamma(\alpha+1)}(\mu - \mu S_n - \beta S_n I_n)$$

Now, we shall apply the same process to the infection equation.

$$\frac{d^\alpha I}{dt^\alpha} = \beta S(r[\frac{t}{r}])I(r[\frac{t}{r}]) - (\mu+\gamma)I(r([\frac{t}{r}])$$

Above is our original equation for infection equation. We'll make our $I(0) = I_0$ and $S(0) = S_0$. We will use a step method having t $\in$ [0,r) then $\frac{t}{r} \in$ [0,1). This would get us a solution of

$$\frac{d^\alpha I}{dt^\alpha} = \beta S_0 I_0 - (\mu+\gamma)I_0$$

So how can we use this differential equation into our initial infection rate equation?

$$I_1(t) - I_0 = I_r^\alpha(\beta S_0 I_0 - (\mu+\gamma)I_0)$$

$$= I_0 + (\beta S_0 I_0 - (\mu+\gamma)I_0)\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)ds}$$

$$= I_0 + (\beta S_0 I_0 - (\mu+\gamma) I_0)\frac{t^\alpha}{\Gamma(1+\alpha)}$$

We can use this same method for I_2. In this case we will let t $\in [r, 2r)$ and $\frac{t}{r} \in [1, 2)$

$$I_2(t) = I_0(r) + \int_r^t (\beta S(r) I(r) - (\mu+\gamma) I(r))$$

$$I_1(r) + (\beta S(r) I(r) - (\mu+\gamma) I(r)) \int_r^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds$$

$$= I_1(r) + (\beta S(r) I(r) - (\mu+\gamma) I(r))\frac{(t-r)^\alpha}{\Gamma(1+\alpha)}$$

We can continue repeating this process to deduce it to a final solution. We will have $t \in [nr, (n+1)r)$ and $\frac{t}{r} \in$ [n,n+1)

$$I_{n+1}(t) = I_n(nr) + nr(\beta SI - (\mu+\gamma) I)\frac{(t-(nr))^\alpha}{\Gamma(1+\alpha)}$$

We will have $t \to (n+1)r$ to obtain the discretization below

$$I_{n+1}((n+1)r) = I_0(nr) + \frac{(((n+1)r) - (nr))^\alpha}{\Gamma(\alpha+1)}(\beta S(nr) I(nr) - (\mu+\gamma) I(nr))$$

That is

$$I_{n+1} = I_n + \frac{r^\alpha}{\Gamma(\alpha+1)}(\beta S_n I_n - (\mu+\gamma) I_n)$$

Since we have acquired all three discretize equations, we simplified our initial SIR Model equations. To analyze the models of the three previous equations we have to assume S(0), I(0), R(0) ≥ 0, S(0)+ I(0) + R(0) = 1.

$$S_{n+1} = S_n + \frac{r^\alpha}{\Gamma(\alpha+1)}(\mu - \mu S_n - \beta S_n I_n)$$

$$I_{n+1} = I_n + \frac{r^\alpha}{\Gamma(\alpha+1)}(\beta S_n I_n - (\mu+\gamma) I_n)$$

$$R_{n+1} = R_n + \frac{r^\alpha}{\Gamma(\alpha+1)}(\gamma I_n - \mu R_n)$$

The solutions for these models have to be non-negative and have initial values satisfying the conditions stated previously. We can guarantee this if these two inequalities hold

$$\frac{r^\alpha}{\Gamma(1+\alpha)}(\beta+\mu) < 1$$

$$\frac{r^\alpha}{\Gamma(1+\alpha)}(\mu+\gamma) < 1$$

We can prove that these inequalities hold by substituting them in the three discretized equations. First we will prove this inequality for S_{n+1}

Theorem 0.1. *If* $\frac{r^\alpha}{\Gamma(1+\alpha)}(\beta+\mu) < 1$ *then* $S_{n+1} > 0 \ \forall n$

Proof. By Direct Proof and Induction
Induction Step: Assume $S_n \geq 0, I_n \geq 0, R_n \geq 0$ and $S_n + I_n + R_n = 1$ Let's observe the equation for S_{n+1}

$$S_{n+1} = S_n + \frac{r^\alpha}{\Gamma(\alpha+1)}(\mu - \mu S_n - \beta S_n I_n)$$

Factor out S_n from this equation to get

$$S_{n+1} = S_n[1 - \frac{r^\alpha}{\Gamma(\alpha+1)}\mu - \frac{r^\alpha}{\Gamma(\alpha+1)}(\beta I_n)] + \frac{r^\alpha}{\Gamma(\alpha+1)}\mu$$

We need to ensure that $[1-\frac{r^\alpha}{\Gamma(1+\alpha)}(\beta I_n+\mu) > 0]$. However we can transform this $[\frac{r^\alpha}{\Gamma(1+\alpha)}(\beta I_n+\mu) < 1]$. Since we know for all n that $\mathrm{S}_n, I_n, R_n \geq 0$ and $S_n+I_n+R_n = 1$, then $0 \leq \mathrm{I} \leq 1$. This means that $\frac{r^\alpha}{\Gamma(1+\alpha)}(\beta I_n+\mu) \leq \frac{r^\alpha}{\Gamma(1+\alpha)}(\beta(1)+\mu)$. So if we consider $\frac{r^\alpha}{\Gamma(1+\alpha)}(\beta+\mu) < 1$ then we can deduce that $\frac{r^\alpha}{\Gamma(1+\alpha)}(\beta I_n+\mu) \leq \frac{r^\alpha}{\Gamma(1+\alpha)}(\beta+\mu)$. Thus we can conclude that 1 - $\frac{r^\alpha}{\Gamma(1+\alpha)}(\beta I_n+\mu) > 0$ as desired. □

Now we can observe the same phenomena for I_n

Theorem 0.2. *If* $\frac{r^\alpha}{\Gamma(1+\alpha)}(\gamma+\mu) < 1$ *then* $I_{n+1} >$ *0* $\forall n$

Proof. By Direct Proof and induction
Induction Step: Assume $S_n \geq 0, I_n \geq 0, R_n \geq 0$ and $S_n + I_n + R_n = 1$. Let's observe the equation for I_{n+1}

$$I_{n+1} = I_n + \frac{r^\alpha}{\Gamma(\alpha+1)}(\beta S_n I_n - (\mu+\gamma)I_n)$$

Factor out I_n from this equation to get

$$I_{n+1} = I_n[1 + \frac{r^\alpha}{\Gamma(\alpha+1)}(\beta S_n - (\mu+\gamma))] =$$

$$I_n[1 - \frac{r^\alpha}{\Gamma(\alpha+1)}(\mu+\gamma)] + \frac{r^\alpha}{\Gamma(\alpha+1)}(\beta S_n)$$

If we want to make sure for all n's that $I_n \geq 0$, we need to ensure that $[1 - \frac{r^\alpha}{\Gamma(\alpha+1)}(\mu+\gamma)] > 0$. Due to the property of $\mathrm{I}_n \geq 0$, we can determine the same for $I_{n+1} \geq 0$ due to induction. So we have to ensure that $I_{n+1} \geq 0$. Since $S_n \geq 0, I_n \geq 0, R_n \geq 0$ and $S_n + I_n + R_n = 1$ then $0 \leq I_n \leq 1$ meaning it will always be positive. If we evaluate the inequality $[1 - \frac{r^\alpha}{\Gamma(\alpha+1)}(\mu+\gamma)] > 0$ we can re write it as $1 > \frac{r^\alpha}{\Gamma(\alpha+1)}(\mu+\gamma)$. Now this has to be true because this would ensure that the inequality of $I_{n+1} \geq 0$. Meaning the inequality of $1 > \frac{r^\alpha}{\Gamma(\alpha+1)}(\mu+\gamma)$ holds as desired.

□

Now we can observe the same phenomena R_n

Theorem 0.3. *If* $\frac{r^\alpha}{\Gamma(1+\alpha)}(\mu+\gamma) < 1$ *then* $R_{n+1} > 0 \; \forall n$

Proof. By Direct Proof and Induction
Induction Step: Assume $S_n \geq 0, I_n \geq 0, R_n \geq 0$ and $S_n + I_n + R_n = 1$. Let's observe the equation of R_{n+1}

$$R_{n+1} = R_n + \frac{r^\alpha}{\Gamma(\alpha+1)}(\gamma I_n - \mu R_n)$$

First we will factor out R_n from the equation

$$R_{n+1} = R_n[1 - \frac{r^\alpha}{\Gamma(\alpha+1)}\mu)] + \frac{r^\alpha}{\Gamma(\alpha+1)}(\frac{\gamma I_n}{R_n})$$

If we want to make sure for all n's that $R_n \geq 0$, we need to ensure that $[1 - \frac{r^\alpha}{\Gamma(\alpha+1)}(\mu+\gamma)] > 0$. Due the property $R_n \geq 0$ we can determine the same for $R_{n+1} \geq 0$ due to induction. If we evaluate the inequality $[1 - \frac{r^\alpha}{\Gamma(\alpha+1)}(\mu+\gamma)] > 0$ we can re write it as $1 > \frac{r^\alpha}{\Gamma(\alpha+1)}(\mu+\gamma)$. In the case of this equation this inequality is $1 > \frac{r^\alpha}{\Gamma(\alpha+1)}(\mu)$. Now this has to be true because this would ensure that the inequality of $\mathrm{R}_{n+1} \geq 0$. Meaning the inequality of $1 > \frac{r^\alpha}{\Gamma(\alpha+1)}(\mu+\gamma)$ holds as desired. □

The two inequalities are demands for the models of three equations. $\frac{r^\alpha}{\Gamma(1+\alpha)}(\beta + \mu) < 1$ illustrates that the susceptible population in percentage who get infected or die is less than one within a unit time. $\frac{r^\alpha}{\Gamma(1+\alpha)}(\mu+\gamma) < 1$ illustrates that the infected population in percentage who get recovered or die is less than one within a unit time.

Since we are using discretized SIR Models to observe Influenza A and its effects towards the population, $\Re_0$ is the essential basic reproductive number signifying the average number of secondary cases of infected individuals caused by a single infected individual in the span of their infected lifetime. $\Re_0$ offers a concept in which it can be used as a medium to predict if an epidemic can either be ceased to exist or continue to persist to affect the population. We can calculate the basic reproductive number, $\Re_0$, for the discretized of SIR Models below

$$S_{n+1} = S_n + \frac{r^\alpha}{\Gamma(\alpha+1)}(\mu - \mu S_n - \beta S_n I_n)$$

$$I_{n+1} = I_n + \frac{r^\alpha}{\Gamma(\alpha+1)}(\beta S_n I_n - (\mu+\gamma) I_n)$$

$$R_{n+1} = R_n + \frac{r^\alpha}{\Gamma(\alpha+1)}(\gamma I_n - \mu R_n)$$

which is given by $\Re_0 = \frac{\beta}{\mu+\gamma}$.

Why is $\Re_0 = \frac{\beta}{\mu+\gamma}$? Here is the justification:
An epidemic starts to occur when the rate of infected persons increases over time, which is $\frac{dI}{dt} > 0$. For our classical SIR model, $\frac{dI}{dt} = \beta SI - \gamma I > 0$ where solving this inequality yields $\beta SI > I$. At an outset of an epidemic where $S = 1$, this guarantees the epidemic that occurs where the inequality is $\frac{\beta}{\gamma} I > I$ which simplifies to $\frac{\beta}{\gamma} > 1$. This signifies that this is the start of the infection and the number of infected persons will increase over time, which results to an epidemic. We now define $\Re_0 = \frac{\beta}{\gamma}$ to represent an epidemic.

Now, we can apply the same concept to our discretized SIR models using I_{n+1}:

$$I_{n+1} = I_n + \frac{r^\alpha}{\Gamma(\alpha+1)}(\beta S_n I_n - (\mu+\gamma) I_n)$$

$$= I_n[1 + \frac{r^\alpha}{\Gamma(\alpha+1)}(\beta S_n - (\mu+\gamma))]$$

An epidemic will occur if $I_{n+1} > I_n$ and this will happen if $\frac{r^\alpha}{\Gamma(\alpha+1)}(\beta S_n - (\mu+\gamma)) > 0$ but since $r > 0$ and $\Gamma(\alpha+1) > 0$, then the inequality above will be true if $(\beta S_n - (\mu+\gamma)) > 0$. To check if this is true, we shall see if we can derive $\Re_0$ where $\Re_0 > 1$

$$(\beta S_n I_n - (\mu+\gamma) I_n) > 0$$

$$I_n(\beta S_n - (\mu+\gamma)) > 0$$

divide both sides to eliminate I_n and now we acquire

$$\beta S_n - (\mu+\gamma) > 0$$

$$\beta S_n > (\mu+\gamma)$$

divide both sides by $(\mu+\gamma)$ to obtain

$$\frac{\beta S_n}{(\mu+\gamma)} > 1$$

If we outset $S_n = 1$, then we get

$$\frac{\beta}{(\mu+\gamma)} > 1$$

and since this satisfies $\Re_0 > 1$, we can define our reproductive number (of an epidemic) to be $\Re_0 = \frac{\beta}{(\mu+\gamma)}$.

By knowing this, we will analyze the tai. and local stability of the discretized SIR Model which depends on $\Re_0$.

Equilibria and Local Stability

We are analyzing equilibria and local stability because we are checking to see if the equilibrium solutions are stable or unstable.

To begin, we will rewrite the dicretized SIR Model in terms of $\Re_0$ which produces the following: The only equation in terms of $\Re_0$ is the infected equation, I_{n+1}, because with β being the rate of transmission, $\Re_0$ replaces it because it represents the number of average secondary cases of infected individuals being infected by a single infected individual through that infected individual's lifespan. In essence, the infection rate equation in terms of $\Re_0$ shows the number of individuals infected from other infected individuals in an infected population.

$$S_{n+1} = S_n + \frac{r^\alpha}{\Gamma(\alpha+1)}(\mu - \mu S_n - \beta S_n I_n)$$

$$I_{n+1} = I_n + \frac{r^\alpha}{\Gamma(\alpha+1)}(\mu+\gamma)(\Re_0 S_n I_n - I_n)$$

$$R_{n+1} = R_n + \frac{r^\alpha}{\Gamma(\alpha+1)}(\gamma I_n - \mu R_n)$$

The equilibria of the new SIR Model in terms of $\Re_0$ must satisfy the following algebraic equations:
$4(\mu - \mu S_n - \beta S_n I_n) = 0$
$(\Re_0 S_n I_n - I_n) = 0$
$(\gamma I_n - \mu R_n) = 0$

these three equations have three unknowns which are S_n, I_n, and R_n, where $\frac{r^\alpha}{\Gamma(\alpha+1)} > 0$

The above equations must be true because we want

$$S_{n+1} - S_n = 0$$

$$I_{n+1} - I_n = 0$$

$$R_{n+1} - R_n = 0$$

to be in the equilibria.

Since we have the new SIR Model in terms of $\Re_0$, we can identify that there are two equilibria, one being the disease free equilibrium $E^0 = (1,0,0)$ for all parameters values and a unique endemic equilibrium when $\Re_0 > 1$ is given by $E^* = (\frac{1}{\Re_0}, \frac{\mu(\Re_0-1)}{\beta}, \frac{\gamma(\Re_0-1)}{\beta})$

The disease free equilibrium is $E^0 = (1,0,0)$ because when we solve S_n, I_n, and R_n, we get the desired outcome of $(1,0,0)$ as the equilibrium point.

To show that this is true, we shall first solve for I_n

$$(\Re_0 S_n I_n - I_n) = 0$$

$$I_n(\Re_0 S_n - 1) = 0$$

$$I_n = 0$$

By knowing that $I_n = 0$, we can apply this to the other two equations. Now, we solve for S_n

$$(\mu - \mu S_n - \beta S_n I_n) = 0$$

$$(\mu - \mu S_n) = 0$$

$$\mu = \mu S_n$$

$$S_n = 1$$

And lastly, we solve for R_n

$$(\gamma I_n - \mu R_n) = 0$$

$$-\mu R_n = 0$$

$$R_n = 0$$

We can conclude that the equilibrium point $E^0 = (1,0,0)$.

We will repeat the same analysis with the second equilibrium point $E^* = (\frac{1}{\Re_0}, \frac{\mu(\Re_0-1)}{\beta}, \frac{\gamma(\Re_0-1)}{\beta})$

First, we shall solve for S_n

$$(\Re_0 S_n I_n - I_n) = 0$$

$$I_n(\Re_0 S_n - 1) = 0$$

$$\Re_0 S_n = 1$$

$$S_n = \frac{1}{\Re_0}$$

Next, we solve for I_n

$$(\mu - \mu S_n - \beta S_n I_n) = 0$$

since we know that $S_n = \frac{1}{\Re_0}$, we can plug that in to get

$$(\mu - \mu(\frac{1}{\Re_0}) - \beta(\frac{1}{\Re_0})I_n) = 0$$

$$\mu - \mu(\frac{1}{\Re_0}) = \beta(\frac{1}{\Re_0})I_n$$
$$\Re_0\mu - \mu = \beta I_n$$
$$I_n = \frac{\mu(\Re_0 - 1)}{\beta}$$

And lastly, we can solve for R_n

$$(\gamma I_n - \mu R_n) = 0$$

since we know that $I_n = \frac{\mu(\Re_0-1)}{\beta}$, we can plug that in to get

$$\gamma(\frac{\mu(\Re_0 - 1)}{\beta}) - \mu R_n = 0$$

$$\gamma(\frac{\mu(\Re_0 - 1)}{\beta}) = \mu R_n$$

$$R_n = \frac{\gamma(\Re_0 - 1)}{\beta}$$

We conclude that this is the second equilibrium point $E^* = (\frac{1}{\Re_0}, \frac{\mu(\Re_0-1)}{\beta}, \frac{\gamma(\Re_0-1)}{\beta})$

Theorem 1. *The disease free equilibrium E_0 of*

$$S_{n+1} = S_n + \frac{r^\alpha}{\Gamma(\alpha+1)}(\mu - \mu S_n - \beta S_n I_n)$$

$$I_{n+1} = I_n + \frac{r^\alpha}{\Gamma(\alpha+1)}(\beta S_n I_n - (\mu+\gamma)I_n)$$

$$R_{n+1} = R_n + \frac{r^\alpha}{\Gamma(\alpha+1)}(\gamma I_n - \mu R_n)$$

is logically asymptotically stable if $0 < \Re_0 < 1$, *and E^0 is unstable if* $\Re_0 > 1$.

Proof. The local asymptotic stability of E^0 can be investigated by linearization. The Jacobian Matrix for

$$S_{n+1} = S_n + \frac{r^\alpha}{\Gamma(\alpha+1)}(\mu - \mu S_n - \beta S_n I_n)$$

$$I_{n+1} = I_n + \frac{r^\alpha}{\Gamma(\alpha+1)}(\beta S_n I_n - (\mu+\gamma)I_n)$$

$$R_{n+1} = R_n + \frac{r^\alpha}{\Gamma(\alpha+1)}(\gamma I_n - \mu R_n)$$

is given by

$$J = \begin{pmatrix} 1-K(\mu+\beta I) & -K\beta S & 0 \\ K(\mu+\gamma)\Re_0 I & 1+K(\mu+\gamma)(\Re_0 S-1) & 0 \\ 0 & K\gamma & 1-K\mu \end{pmatrix}$$

where $K = \frac{r^{\alpha}}{\Gamma(1+\alpha)}$.
This is true because to set up the Jacobian matrix, we base off each entry by a partial derivative accordingly, see [6]

$$J = \begin{matrix} \frac{\partial S_{n+1}}{\partial S_n} & \frac{\partial S_{n+1}}{\partial I_n} & \frac{\partial S_{n+1}}{\partial R_n} \\ \frac{\partial I_{n+1}}{\partial S_n} & \frac{\partial I_{n+1}}{\partial I_n} & \frac{\partial I_{n+1}}{\partial R_n} \\ \frac{\partial R_{n+1}}{\partial S_n} & \frac{\partial R_{n+1}}{\partial I_n} & \frac{\partial R_{n+1}}{\partial R_n} \end{matrix}$$

where this produces the desired Jacobian matrix

$$J = \begin{pmatrix} 1-K(\mu+\beta I) & -K\beta S & 0 \\ K(\mu+\gamma)\Re_0 I & 1+K(\mu+\gamma)(\Re_0 S-1) & 0 \\ 0 & K\gamma & 1-K\mu \end{pmatrix}$$

The Jacobian matrix evaluated at E^0 is given by

$$J = \begin{pmatrix} 1-Ku & -K\beta & 0 \\ 0 & 1+K(\mu+\gamma)(\Re_0-1) & 0 \\ 0 & K\gamma & 1-K\mu \end{pmatrix}$$

We know this is true because $E^0 = (1, 0, 0)$. So we plug in values respectively for $S_n = 1$, $I_n = 0$, and $R_n = 0$ for each entry.

We look for the necessary and sufficient conditions for E^0 to have all eigenvalues; roots of the characteristic polynomial; less than one in modulus

$$F(\lambda) = \lambda^3 - (A+B+C)\lambda^2 + (AB+AC+BC+K\beta SD)\lambda - ABC - K\beta SDC$$

where $A = 1-K(\mu+BI)$, $B = 1+K(\mu+\gamma)(\Re_0 S-1)$, $C = 1-K\mu$, and $D = K(\mu+\gamma)\Re_0 I$.

The eigenvalues associated to J evaluated at E^0 are $\lambda_{1,2} = 1-K\mu$, and $\lambda_3 = 1+K(\mu+\gamma)(\Re_0-1)$. Now we have $|\lambda_{1,2}| < 1$ if $0 < K\mu < 2$, while $|\lambda_3| < 1$ if $0 < K(\mu+\gamma)(\Re_0-1) < 2$. The condition $\mu+\gamma < 1$ together with $\Re_0 < 1$ guarantees that $|\lambda_i| < 1$, $i = 1, 2, 3$ and hence E^0 is locally asymptotic stable. If $\Re_0 > 1$, we will have $|\lambda_3| > 1$ and hence E^0 is unstable.

□

Theorem 2. *E^* is asymptotically stable if* $\Re_0 > 1$

Proof. Evaluating J at E^* gives

$$J = \begin{pmatrix} 1 - K\mu\Re_0 & \frac{-K\beta}{\Re_0} & 0 \\ 1 + \frac{K\mu}{\beta}(\mu+\gamma)(\Re_0 - 1) & 1 & 0 \\ 0 & K\gamma & 1 - K\mu \end{pmatrix}$$

The roots of the characteristic equation of eigenvalues associated with J evaluated at E^* is

$$F(\lambda) = (1 - K\mu - \lambda)((1 - K\mu\Re_0 - \lambda)(1 - \lambda) + K^2\mu(\mu+\gamma)(\Re_0 - 1))$$

$= \lambda^2 + \lambda(K\mu\Re_0 - 2) + 1 - K\mu\Re_0 + K^2\mu(\mu+\gamma)(\Re_0 - 1)$
with $\lambda_1 = 1 - K\mu$, where $|\lambda_1| < 1$ if $0 < K\mu < 2$.

The constant term in $F(\lambda)$ is given by $C = 1 - K\mu\Re_0 + K^2(\mu+\gamma)(\Re_0 - 1) < 1$. Thus, the criteria is satisfied for the characteristic equation which implies that $|\lambda_2| < 1$ and $|\lambda_3| < 1$. Hence, E^* is asmptotically stable.

□

Numerical Simulations

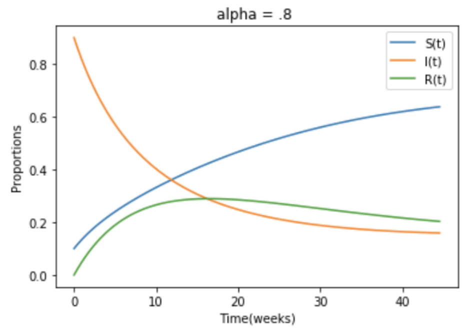
We will perform some numerical simulations to illustrate our analytical results to reveal the complex dynamics of our equations and to see what is the behaviour of our solution:

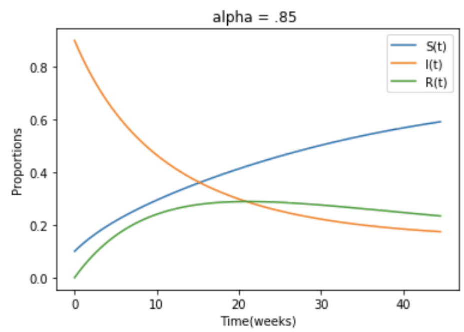
$$S_{n+1} = S_n + \frac{r^\alpha}{\Gamma(\alpha+1)}(\mu - \mu S_n - \beta S_n I_n)$$

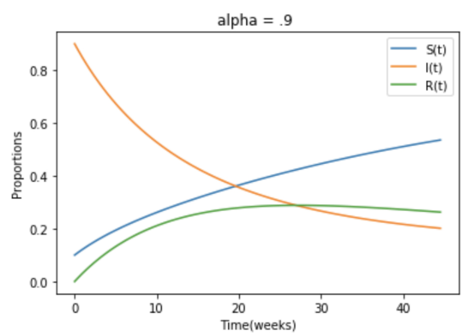
$$I_{n+1} = I_n + \frac{r^\alpha}{\Gamma(\alpha+1)}(\mu+\gamma)(\Re_0 S_n I_n - I_n)$$

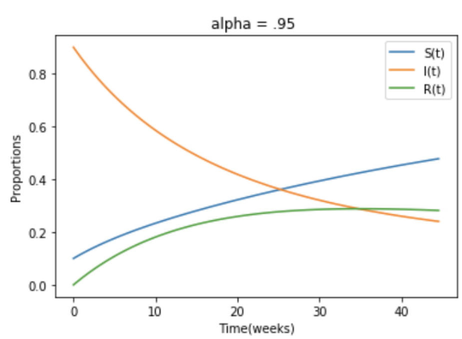
$$R_{n+1} = R_n + \frac{r^\alpha}{\Gamma(\alpha+1)}(\gamma I_n - \mu R_n)$$

In all numerical simulations we take r = 0.01, $\mu = 1$, $\gamma = 1$, and $\beta = 3$. With these parameter values, direct calculations yield $\Re_0 = 1.5$. The graphs shown in figure 2-5 shows the trajectories of the SIR model depending on the changes in α. α = .8,.85,.9,.95 respectively corresponding numerically with figures 2 through 5. When $\alpha \longrightarrow 1$, the number of infected individuals increase while the number of recovered individuals decrease. Theorem 2 imply that $E^0 = (1, 0, 0)$ is locally asymptotically stable when $\Re_0 < 1$ and unstable when $\Re_0 > 1$[8]. On the contrary, $E^* = (0.6667, 0.1667, 0.1667)$ is asymptotically stable when $\Re_0 >$


alpha = .8
S(t)
I(t)
R(t)
Proportions
Time(weeks)


alpha = .85
S(t)
I(t)
R(t)
Proportions
Time(weeks)


alpha = .9
S(t)
I(t)
R(t)
Proportions
Time(weeks)


alpha = .95
S(t)
I(t)
R(t)
Proportions
Time(weeks)

1. So depending on the initial conditions, the system may go to E* Therefore when $\Re_0 < 1$ every person who contracts the disease will infect less than one person before dying or recovering while when $\Re_0 > 1$, there will be a disease outbreak. All solutions with a positive initial conditions are non-negative, then the numerical simulations show no complex dynamics. On the other hand, if the initial conditions are negative then the model has complex dynamics.

Conclusion

Our goal of this paper is to learn the rudimentary principles of fractional differential equations and to verify if solving a classical SIR Model with the use of fractional derivatives is feasible; in other words, can we obtain the results we want based on our solutions? Or can we distinguish which solution is the most accurate pertaining to our data?

The standard way of approaching this is to solve the fractional differential SIR Model and compare it with an integer order SIR Model. Using a discretization we can make fractional order differential equations easier to solve and plot on a graph. By plotting a fractional order set of equations we obtain memory which we couldn't obtain using integer order equations.

Since the standard way of solving SIR Models is too ambitious to achieve because of its difficulty, we shall look at the main components that will determine if our solutions are correct and appropriate with the the data we came up with and with the steps we took to solve the problem.

The fractional SIR Model can be solved by using fractional derivatives of non-integer order.

$$\frac{d^\alpha S}{dt} = \mu - \mu S(t) - \beta S(t)I(t)$$

$$\frac{d^\alpha I}{dt} = \beta S(t)I(t) - (\mu + \gamma)I(t)$$

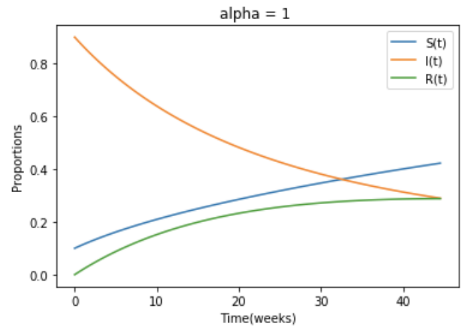
$$\frac{d^\alpha R}{dt} = \gamma I(t) - \mu R(t)$$

But since this is too difficult to solve, we applied the concept of discretization towards are fractional SIR Model which becomes a more manageable task to achieve.

$$S_{n+1} = S_n + \frac{r^\alpha}{\Gamma(\alpha+1)}(\mu - \mu S_n - \beta S_n I_n)$$

$$I_{n+1} = I_n + \frac{r^\alpha}{\Gamma(\alpha+1)}(\beta S_n I_n - (\mu + \gamma)I_n)$$

$$R_{n+1} = R_n + \frac{r^\alpha}{\Gamma(\alpha+1)}(\gamma I_n - \mu R_n)$$



The results when solving the discretized fractional SIR Model will be beneficial when finding equilibria and stability, which will be explained in the latter.

Solving for equilibria and stability will be based on our discretized fractional SIR Model in terms of the reproductive number, $\Re_0$.

$$S_{n+1} = S_n + \frac{r^\alpha}{\Gamma(\alpha+1)}(\mu - \mu S_n - \beta S_n I_n)$$

$$I_{n+1} = I_n + \frac{r^\alpha}{\Gamma(\alpha+1)}(\mu+\gamma)(\Re_0 S_n I_n - I_n)$$

$$R_{n+1} = R_n + \frac{r^\alpha}{\Gamma(\alpha+1)}(\gamma I_n - \mu R_n)$$

We solved a Jacobian matrix when we evaluated for the equilibrium points, $E^0 = (1,0,0)$ and $E^* = (\frac{1}{\Re_0}, \frac{\mu(\Re_0-1)}{\beta}, \frac{\gamma(\Re_0-1)}{\beta})$ under the conditions that $0 < \Re_0 < 1$ for E^0 and $\Re_0 > 1$ for E^*. Knowing this is vital for our plots when we compare a classical SIR Model and a fractional derivative SIR Model later.

With our use of fractional derivatives, we can change our α such that it is not a whole number. This allows us to conclude the behavior of the graphs as α is approaching one, but not at one. As α is increasing from 0.8 to 0.95, the infected population is decreasing at a significantly slower rate essentially increasing in population. The susceptible population is decreasing as α increases. The recovered population increases as α increases.

Using the information given from the plots, we can speculate that the different values for α determines the speed of the rate of each respected function (either fast or slow), S, I, and R, with respect to time. For example: when α is increasing, we can pinpoint that $I(t)$ and $R(t)$ move at almost relatively the same rate to reach the respected equilibrium point, which also shows that these respective functions are asymptotically stable.

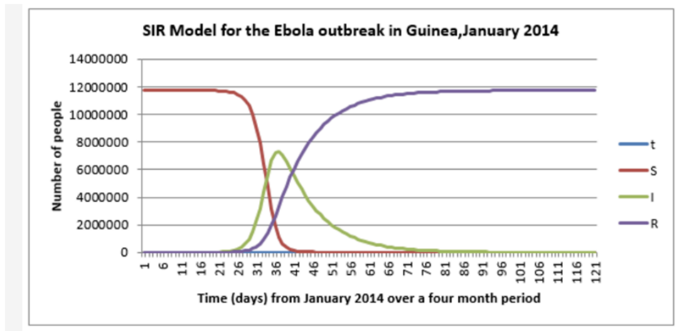


Figure 2: Ebola SIR Model

To compare our equations to a real classical SIR model, we will use the data from the Ebola outbreak in Guinea, see [1]. When trying to determine if the fractional derivative SIR Model is a viable way of modeling an epidemic, we have to compare those plots of the fractional derivative SIR Model with that of a classical SIR model. In comparison to classical SIR Model, our fractional order SIR Model has some discrepancies. In a classical SIR Model, the susceptible population is decreasing as time increases, but for fractional derivative equations the susceptible population is increasing. We speculate that the fractional derivative SIR Models serves a purpose for its inclusion of memory, but through our comparison of the $\alpha = 1$ SIR Model and classical SIR model, the fractional derivative SIR Models is not as accurate. We also take into account that in the $\alpha = 1$ graph, the susceptibility rate gradually increases as time increases but in the classical SIR Model, the susceptibility and infection rates both decrease as time increases. As for the recovery rate, it increases exponentially as time increases, which we think the classical SIR Model is a more accurate and suitable to represent how a population is being affected as time goes by.

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